Unit - I

Section: 2.5

A counting Principle

If H and K are two subgroups of a group G then we define $HK = \{hk \mid h \in H \ \& \ k \in K\}$

Lemma: 2.5.1

HK is a subgroup of G iff $HK = KH$.

Proof:

Suppose that $HK = KH$.

ie) If $h \in H$ and $k \in K$ then $hk = k_1 h_1$ for some $k_1 \in K$ and $h_1 \in H$ (it need not be that $k_1 = k$ or $h_1 = h$)

To prove: HK is a subgroup of G.

Let $x, y \in HK$

Then $x = hk$ & $y = h'k'$

Now $xy = (hk)(h'k')$

$= h(kh')k'$

$= h(h_2k_2)k'$ [$\because kh' \in KH = HK$]

$= (hh_2)(k_2k')$

$\in HK$.

$\therefore xy \in HK$.

Let $x \in HK$. Then $x = hk$

To prove $x^{-1} \in HK$

Now $x^{-1} = (hk)^{-1}$

$= k^{-1}h^{-1}$

$\in KH = HK$.

$\therefore x^{-1} \in HK$.

Hence HK is a subgroup of G

Conversely,

Suppose HK is a subgroup of G

Then for any $h \in H$, $k \in K$ we have $h^{-1}k^{-1} \in HK$.

$\therefore kh = (h^{-1}k^{-1})^{-1} \in HK$

[$\because$ HK is a subgroup]

Since $kh \in KH$, $KH \subseteq HK$ — ①

Let $x \in HK$

Since HK is a subgroup, $x^{-1} \in HK$.

Let $x^{-1} = hk$.

$\therefore (x^{-1})^{-1} = (hk)^{-1}$

$= k^{-1}h^{-1} \in KH$.

$\therefore x \in KH$.

$\therefore HK \subseteq KH$ — ②

From ① & ②, $KH = HK$.

Corollary:

If H, K are subgroups of the abelian group G then HK is the subgroup of G

Proof:

Since G is abelian, $HK = KH$

By Previous lemma: 2.5.1.

HK is a subgroup.

Thm: 2.5.1

If H and K are finite subgroups of G of orders $O(H)$ and $O(K)$ respectively then

$$O(HK) = \frac{O(H)\,O(K)}{O(H \cap K)}$$

Proof:

We Shall prove this theorem in two cases

Case i:

Suppose $H \cap K = \{e\}$ [$\because$ H and K subgroup of G].

$\therefore O(H \cap K) = 1$

$\therefore$ We have prove that $O(HK) = O(H)O(K)$

WKT $HK = \{hk \mid h \in H \& k \in K\}$

We shall prove that the repraserntation of the element of HK is unique.

ie) To prove: if $hk = h_1k_1$ then $h = h_1$ & $k = k_1$

Suppose $hk = h_1k_1$

$\Rightarrow h_1^{-1}(hk) = h_1^{-1}(h_1k_1)$

$\Rightarrow (h_1^{-1}h)k = (h_1^{-1}h_1)k_1$ by also is $h_1^{-1}h_1 = e$

$\Rightarrow (h_1^{-1}h)k = ek_1$

$\Rightarrow (h_1^{-1}h)k = k_1$

$\Rightarrow (h_1^{-1}h)kk^{-1} = k_1k^{-1}$

$\Rightarrow h_1^{-1}h = k_1k^{-1}$

$\Rightarrow h_1^{-1}h \in H \cap K$ and $k_1k^{-1} \in H \cap K$.

$\therefore h_1^{-1}h = e$ and $k_1k^{-1} = e$

Now, $h_1^{-1}h = e$.

$h_1(h_1^{-1}h) = h_1e$

$\Rightarrow h = h_1$, $\text{lly } k = k_1$

$\therefore$ HK contains $O(H)O(K)$ elements

$\therefore O(HK) = O(H)O(K)$.

Case ii:

Suppose $H \cap K \neq \{e\}$

In this case we shall prove that every element of HK is repeated at ~~most~~ least $O(H \cap K)$ times.

If $h_1 \in H \cap K$, then we have $hk = h(h_1 h_1^{-1})k$, $h \in H$ and $k \in K$

ie $hk = (hh_1)(h_1^{-1}k)$

ie $hk \in HK$ [$\because h_1 \in H \cap K$ ie) $h_1 \in H$ & $h_1 \in K$

$\therefore hh_1 \in H$ & $h_1^{-1}k \in K$].

Claim: Every element of HK is duplicated atleast $O(H \cap K)$ times

Suppose $hk = h'k'$ for some $h, h' \in H$ & $k, k' \in K$.

$\Rightarrow h^{-1}(hk) = h^{-1}(h'k')$

$\Rightarrow k = (h^{-1}h')k'$

$\Rightarrow kk'^{-1} = (h^{-1}h')k'k'^{-1}$

$\Rightarrow kk'^{-1} = h^{-1}h' = u$ (say)

$\therefore u \in H \cap K$

Consider $h^{-1}h' = u$.

$\Rightarrow h(h^{-1}h') = hu$

$\Rightarrow h' = hu$ ——— ①

u^{ly} $kk'^{-1} = u$

$\Rightarrow (kk'^{-1})^{-1} = u^{-1}$

$\Rightarrow k'k^{-1} = u^{-1}$

$\Rightarrow (k'k^{-1})k = u^{-1}k$

$\Rightarrow k' = u^{-1}k$ ——— ②

① & ② $\Rightarrow$ $h'k' = (hu)(u^{-1}k)$

$h'k' = hk$

Thus every element of HK is duplicated ~~atmost~~ atleast $O(H \cap K)$ times

$\therefore$ The no of distinct elements of HK is

$\frac{O(H)O(K)}{O(H \cap K)}$

ie) $O(HK) = \dfrac{O(H)O(K)}{O(H\cap K)}$

Hence the theorem.

Corollary:

If H and K are subgroups of G and $O(H) > \sqrt{O(G)}$, $O(K) > \sqrt{O(G)}$ then $H\cap K \neq (e)$ $\{e\}$

Proof

Since H and K are subgroups of G

We have $HK \subseteq G$.

$\therefore O(HK) \leq O(G)$

Now. $O(G) \geq O(HK)$

$$= \frac{O(H)\,O(K)}{O(H\cap K)}$$

$$> \frac{\sqrt{O(G)}\,\sqrt{O(G)}}{O(H\cap K)}$$

$$= \frac{O(G)}{O(H\cap K)}.$$

$\therefore O(G) > \dfrac{O(G)}{O(H\cap K)}$.

$\therefore O(H\cap K) > 1$

$\therefore H\cap K \neq (e)$

Section 2.6

Normal Subgroups and Quotient groups

Defn:

A subgroup N of G is said to be a normal subgroup of G if for every $g\in G$ and $n\in N$ such that $gng^{-1}\in N$

Equivalently we mean that the set of all $gng^{-1} \in N$ then N is a normal subgroup of G iff $gNg^{-1} \subset N$ for every $g \in G$

Eg:

Let $A = \{1, 2, 3\}$. $S_3 = \{e, \rho_1, \rho_2, \ldots, \rho_5\}$ where (permutations)

$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$; $\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$; $\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$\rho_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$; $\rho_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$; $\rho_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

Let $H = \{e, \rho_1, \rho_2\}$ which is a subgroup of S_3 and H is the normal subgroup of S_3

Lemma: 2.6.1

N is a normal subgroup of G iff $gNg^{-1} = N$ for every $g \in G$

Proof:

Suppose $gNg^{-1} = N$

Then $gNg^{-1} \subset N$

$\therefore N$ is a normal subgroup of G

Conversely, Suppose N is a normal subgroup of G

Then $gNg^{-1} \subset N$ for every $g \in G$ —— (1)

Now $g^{-1}Ng = g^{-1}N(g^{-1})^{-1} \subset N$

$\therefore g^{-1}Ng \subset N$ —— (2)

Now $N = (gg^{-1})N(gg^{-1})$

$= g(g^{-1}Ng)g^{-1}$ [by associative law]

$\subset gNg^{-1}$ [$\because$ by (2)]

$\therefore N \subset gNg^{-1}$ —— ③

From ① & ③ $gNg^{-1} = N$

Lemma : 2.6.2

A subgroup N of G is a normal subgroup of G iff every left coset of N in G is a right coset of N in G

Proof.

Suppose every left coset of N in G is a right coset of N in G.

Let $g \in G$

Then $g = ge \in gN$

Also $g = eg \in Ng$

Given $gN = Ng$

$\Rightarrow gNg^{-1} = N\underbrace{gg^{-1}}_{e}$

$\Rightarrow gNg^{-1} = N$

$\Rightarrow$ N is a normal subgroup of G by previous lemma 2.6.1

Conversely.

Suppose N is a normal subgroup of G.

TP $gN = Ng$ for all $g \in G$.

Then $gNg^{-1} = N$ [by lemma 2.6.1].

$\Rightarrow (gNg^{-1})g = Ng$.

$\Rightarrow gN = Ng$

Lemma: 2.6.3.

A subgroup N of G is a normal subgroup of G iff the product of two right cosets of N in G is

$NaNb = Nab$

again a right coset of N in G.

Proof:

Suppose N is a normal subgroup of G

Let Na and Nb be two right cosets of N in

To prove: $NaNb = Nab$

Now $NaNb = N(aN)b$

$= N(Na)b$ [∵ by lemm 2 b.

$= NNab$

$= Nab$.

Conversely, suppose the product of two right coset of N in G is again a right coset of N in G.

T.P N is normal in G

We have $ea \in Na$ and $eb \in Nb$

$\therefore (ea)(eb) \in NaNb$

$= Nab$

$\Rightarrow ab \in Nab$.

We have $NaNb = Nab$

Let $x \in NaNb$

Then $x = n_1 a n_2 b$ for some $n_1, n_2 \in N$

Since $x \in Nab$ we have $x = n_3 ab$ for some $n_3 \in N$

$\therefore n_1 a n_2 b = n_3 ab$

$\Rightarrow n_1 a n_2 = n_3 a$ [by right Cancellation law]

$\Rightarrow (n_1 a n_2) a^{-1} = n_3 a a^{-1}$

$\Rightarrow n_1 a n_2 a^{-1} = n_3$

$\Rightarrow n_1^{-1} n_1 a n_2 a^{-1} = n_1^{-1} n_3$

$\Rightarrow a n_2 a^{-1} = n_1^{-1} n_3 \in N$

$\Rightarrow a n_2 a^{-1} \in N \;;\; n_2 \in N$

$\therefore N$ is normal in G

Hence proved

Defn

Let N be a normal subgroup of a group G. Let G/N denote the set of all right cosets of N in G. Then G/N is also a group which is called the quotient group or factor group ie) $G/N = \{Na / a \in G\}$

Thm: 2.6.1

G/N is a group

Proof:

Let $Na, Nb \in G/N$

Then $NaNb = Nab$ [$\because$ by lemma 2.6.3]

$\in G/N$

Let $Na, Nb, Nc \in G/N$

$$
\begin{aligned}
Na(NbNc) &= NaNbc \\
&= Na(bc) \\
&= N(ab)c \\
&= NabNc \\
&= (NaNb)Nc
\end{aligned}
$$

$N = Ne \in G/N$

$\therefore NaNe = Nae$
$= Na$

Also $NeNa = Nea$
$= Na$

$\therefore N \in G/N$ is the identity element.

Let $Na \in G/N$

Now, $NaNa^{-1} = Naa^{-1}$
$= Ne = N$

$\therefore Na^{-1}$ is the inverse of Na, $\therefore G/N$ is a group

Result: 1

If N is a normal subgroup of G which is abelian then G/N is also abelian

Proof

We know that G/N is a group [∵ by previous thm]

Let $Na, Nb \in G/N$

Now, $NaNb = Nab$

$= Nba$ [∵ $a, b \in G$, G is an abelian $ab = ba$].

$= NbNa$

∴ G/N is an abelian.

Result: 2.

Since G/N has the set of all the right cosets of N in G

We have $o(G/N) = i_G(N)$

$$= \frac{o(G)}{o(N)}$$

Section: 2.7.

Homomorphism

Defn.

A mapping φ from a group G into a group $\bar{G}$ is said to be a homomorphism if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$.

Eg

1. $\varphi(x) = e$ for all $x \in G$

This is trivially a homomorphism. Also $\varphi(x) = x$ for every $x \in G$ is a homomorphism.

2. Let G be the group of all real number under addition, Let $\bar{G}$ be the group of non-zero real numbers under ordinary multiplication.

Define $\varphi : G \to \bar{G}$ by $\varphi(a) = 2^a$ is a homomorphism

For Let $a, b \in G$.